Tight Bound for Sampling q-coloring
via Coupling from the Past

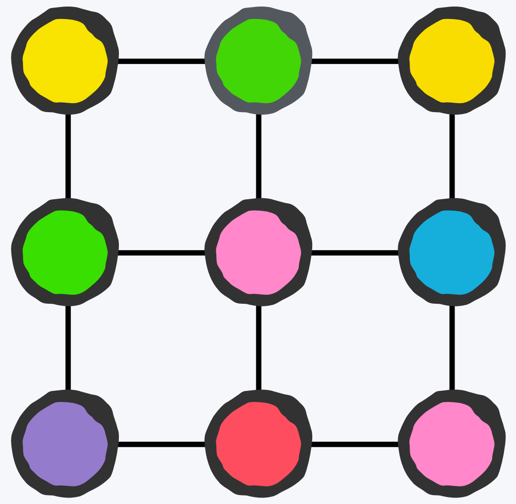
Tianxing Ding[†] · Hongyang Liu[†] · Yitong Yin[†] · Can Zhou[†]

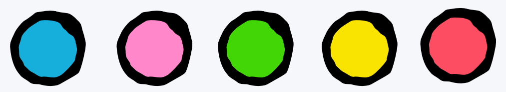
The talker

[†] Nanjing University

Graph Coloring & Sampling

Graph Coloring



$[q] = \{$  $\}$

Input : $G = (V, E)$ with max degree Δ, colors $[q]$.

Proper coloring $\sigma : V \to [q]$

$$\Omega = \{\sigma : \forall (u, v) \in E, \sigma(u) \neq \sigma(v)\}$$

Output : An arbitrary $\sigma \in \Omega$.

Sampling Graph Coloring

Input : $G = (V, E)$ with max degree Δ, colors $[q]$. $(q > \Delta)$

Uniform Distribution $\mu = \text{Uniform}(\Omega)$

Output : $\sigma \in \Omega$ form distribution μ^* such that

$d_{\text{TV}}(\mu, \mu^*) \leq \epsilon$ (approximate)

$\mu = \mu^*$ (perfect)

Glauber dynamic

Glauber dynamic

Start with arbitrary proper coloring σ;

For each $1 \le t \le T$:

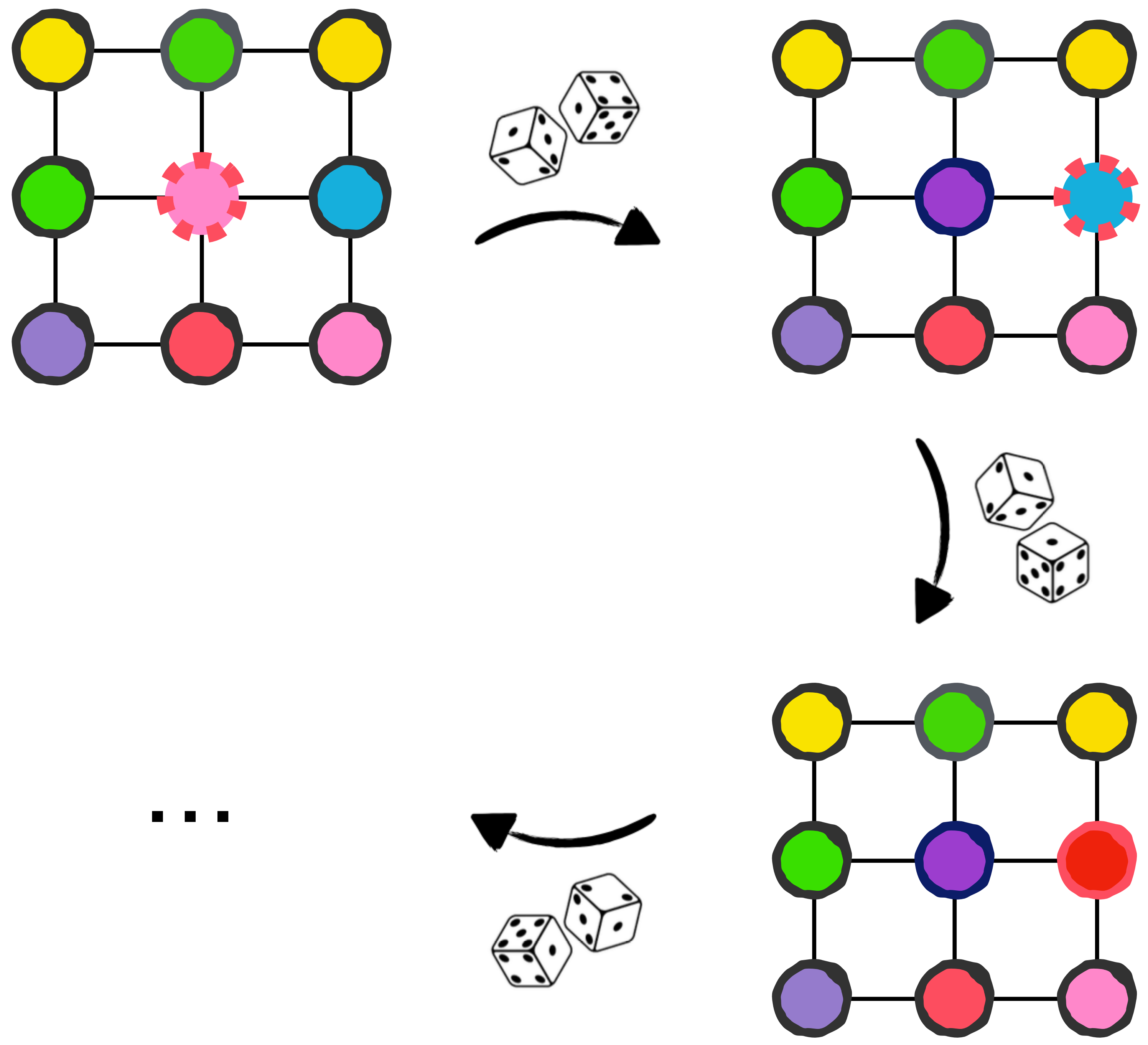
Pick $v \sim \mathrm{Uniform}(V)$;

Resample $\sigma(v) \sim \mu_v\left(\cdot \mid \sigma_{V\setminus\{v\}}\right)$;

$\mathrm{Uniform}([q]\setminus\sigma(\Gamma(v)))$

Output σ;

Glauber update



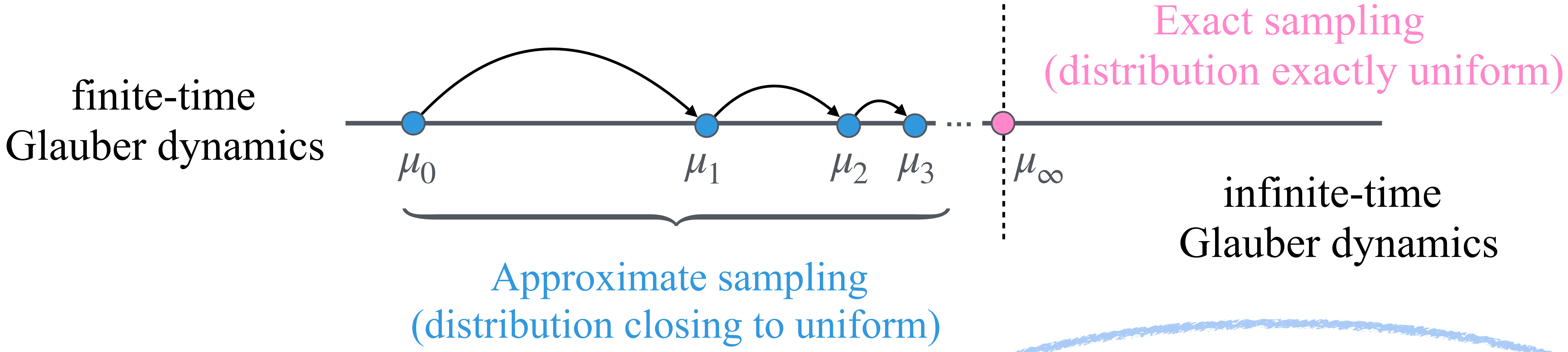
Stationary distribution is μ

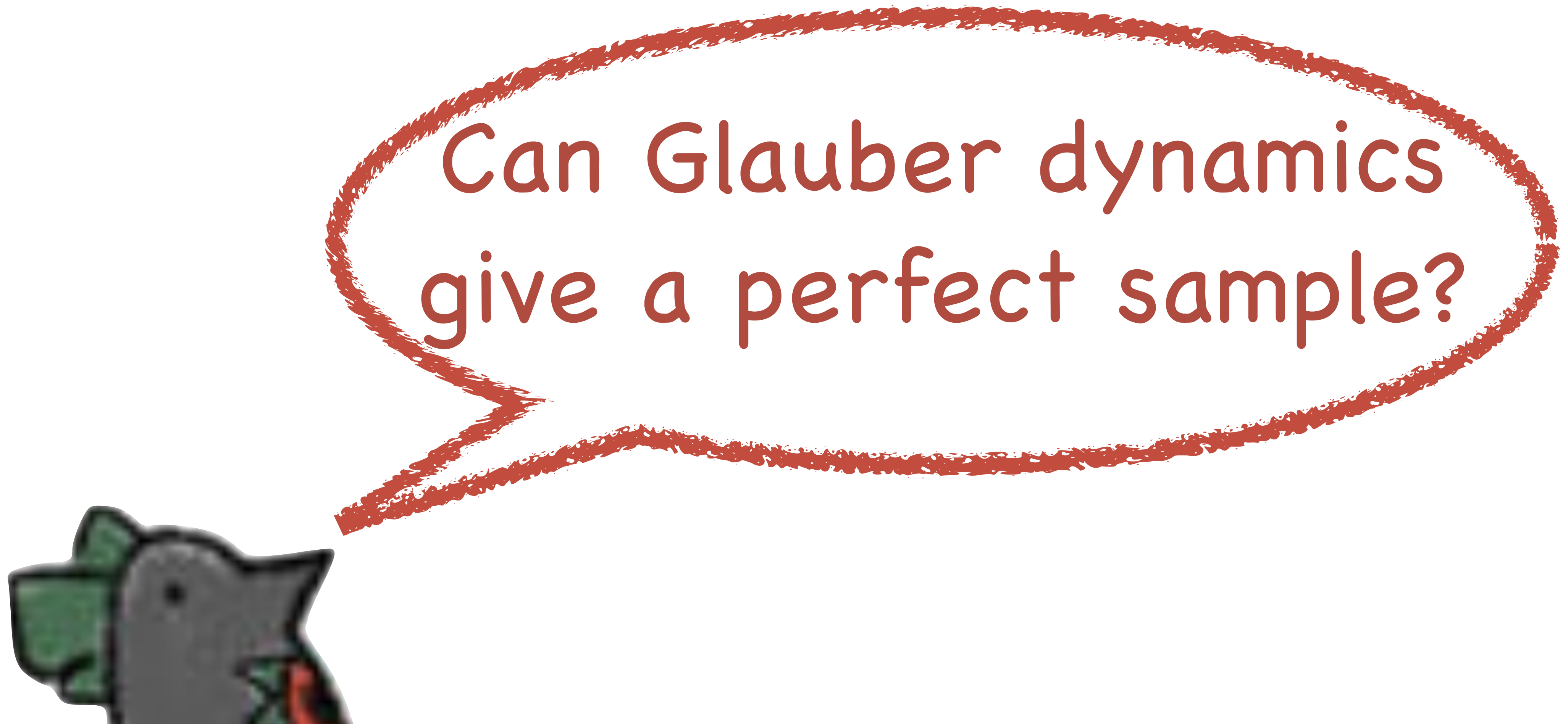
Simulating can get approximate sampling

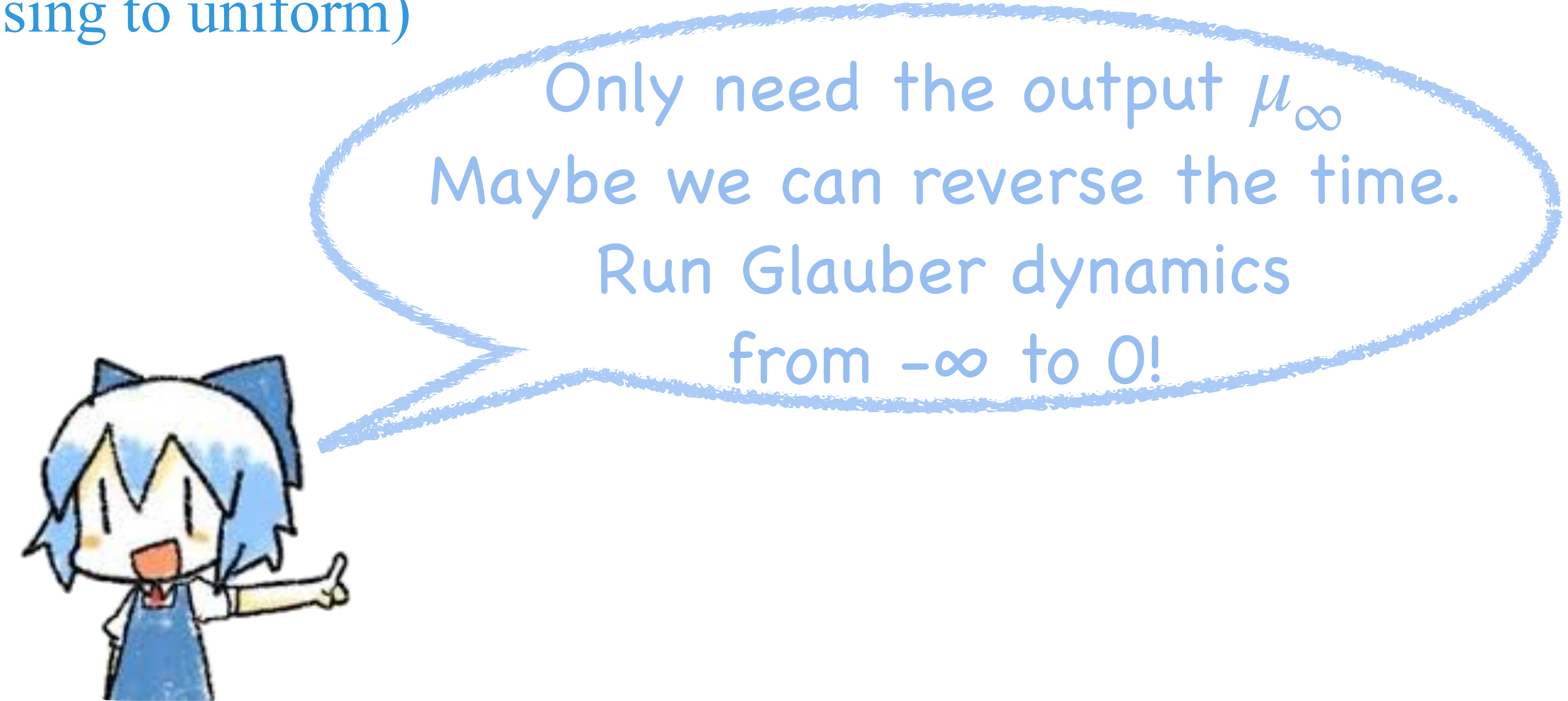
Recent rapid-mixing results reach $q \ge 1.809\Delta$ [CV24]

How exact stationary

Exact Sampling vs. Approximate Sampling

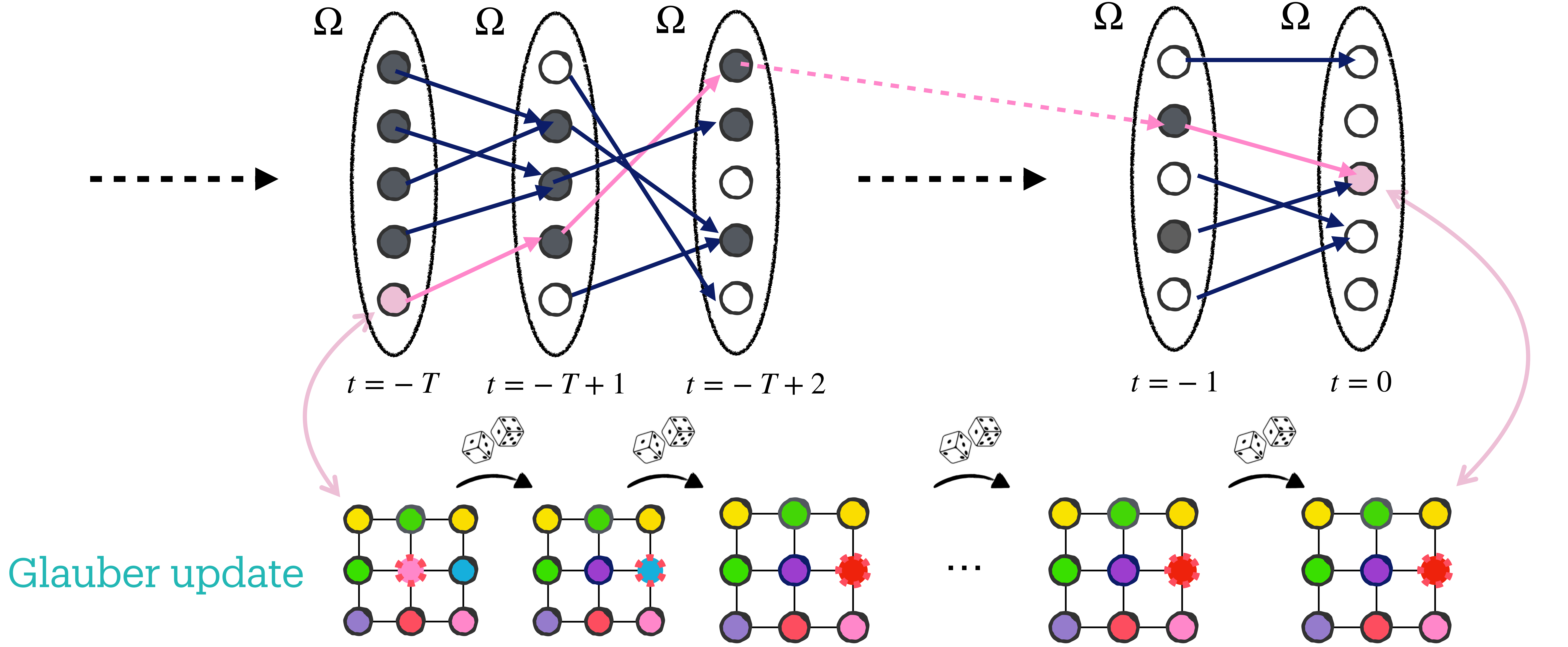






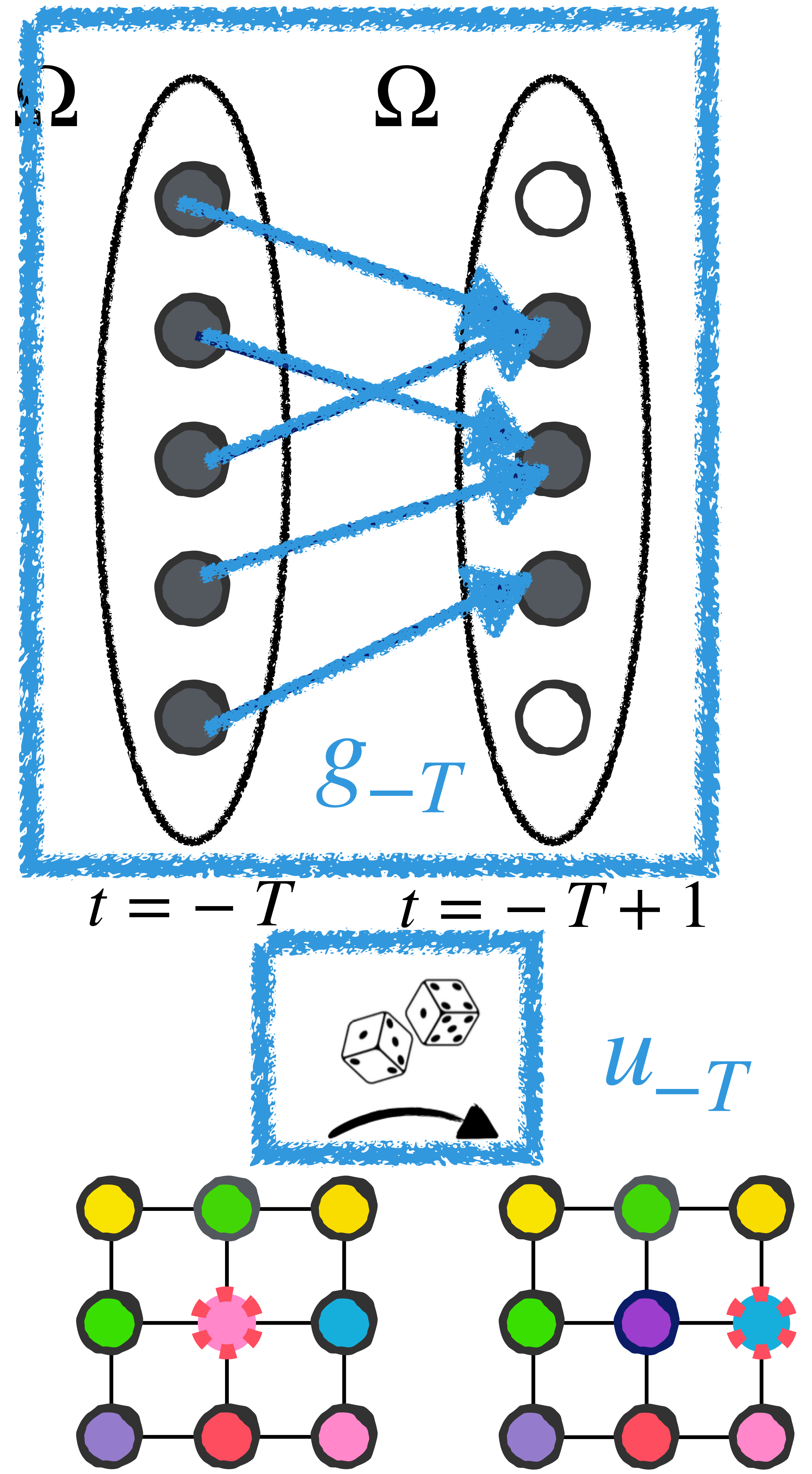
Exact sampling via Coupling from the past

Simulate Glauber Dynamic from $t = -\infty$.



Formalizing CFTP via Grand Coupling

g_{-t} : Random map determined by u_{-T}



Grand Coupling

Glauber update with shared randomness.

$$g : \Omega \times [0,1] \to \Omega$$

such that $\Pr_{U \sim \mathrm{Uniform}[0,1]} [g(x, U) = y] = p(x, y)$.

▲Kernel of Glauber dynamic

Coupling from the past

For $t = -1$ **to** $-\infty$:

Construct g_t and generate u_t;

$G \leftarrow G \circ g(\cdot, u_t)$;

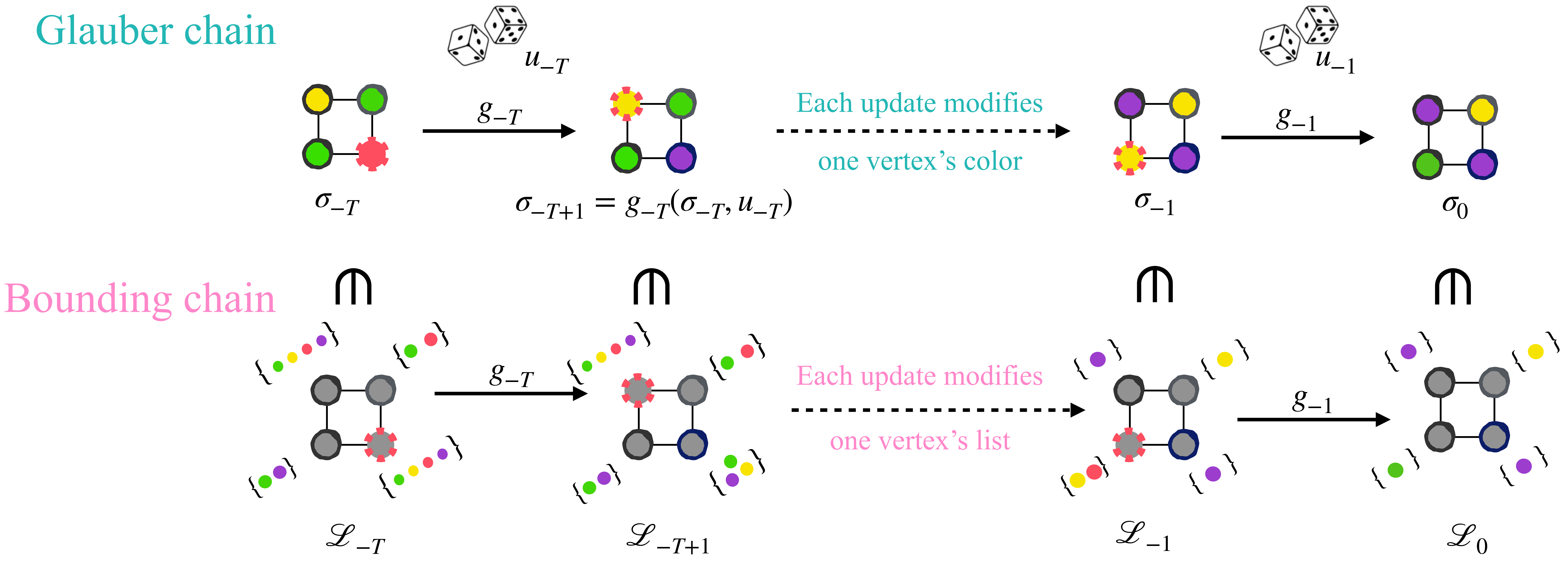
Output $G(\cdot)$ if $\Phi(G) = \mathrm{True}$;

sufficient condition for proving G is constant

computational hard

Bounding Chain

Bounding chain is a Markov chain over $2^{[q]^V}$



$\mathrm{Img}(g_{-T} \circ g_{-T+1} \circ \cdots \circ g_{-t}) \subseteq \mathscr{L}_{-t}$

$\boxed{|\mathscr{L}_0| = 1 \Rightarrow G \text{ is constant function;}}$

Our results

Tight bound for bounding-chain CFTP

Upper Bound

A bounding-chain CFTP algorithm for sampling uniform q-colorings when

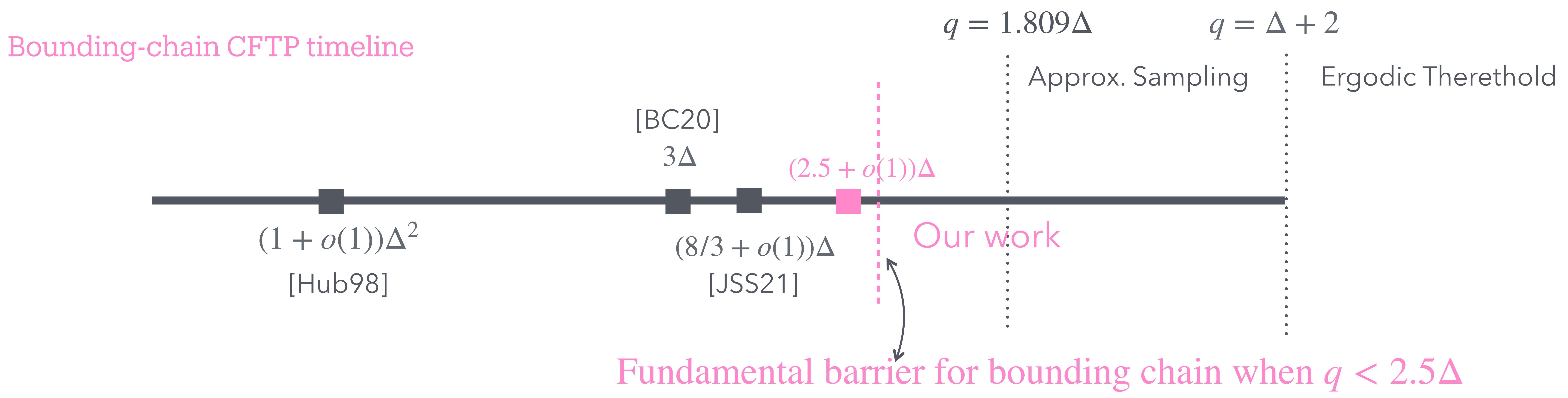
$$q \geq 2.5\Delta + 2\sqrt{\log(\Delta + 1)\Delta}$$

In expected time $\tilde{O}(n\Delta^2)$.

Lower Bound

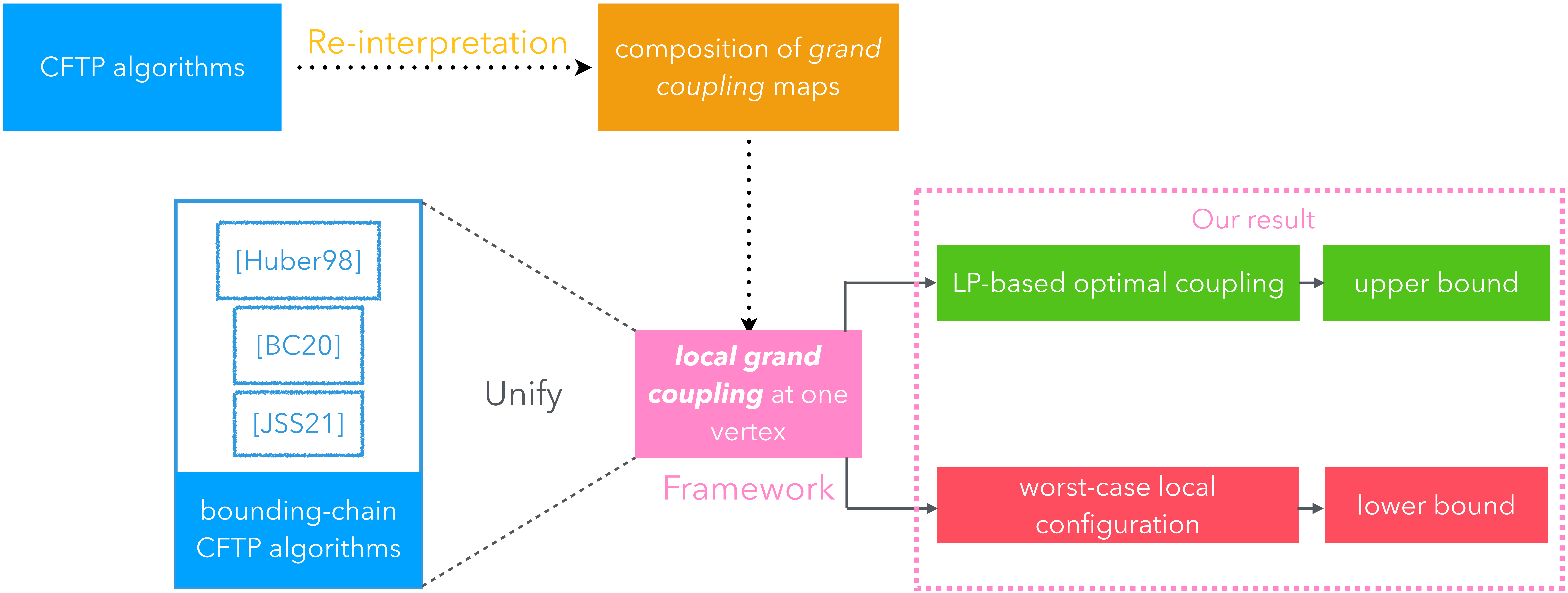
No contractive bounding-chain CFTP algorithm can go below

$$q \leq 2.5\Delta - O(1)$$



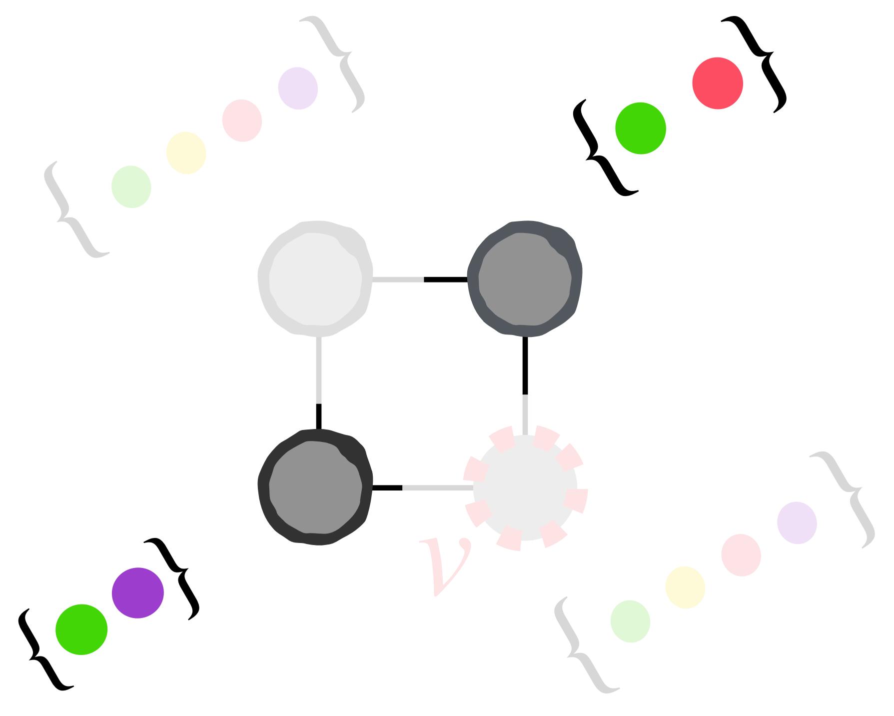
Our results

Grand Coupling as Framework for CFTP



Local Grand Coupling : An Example

Update v



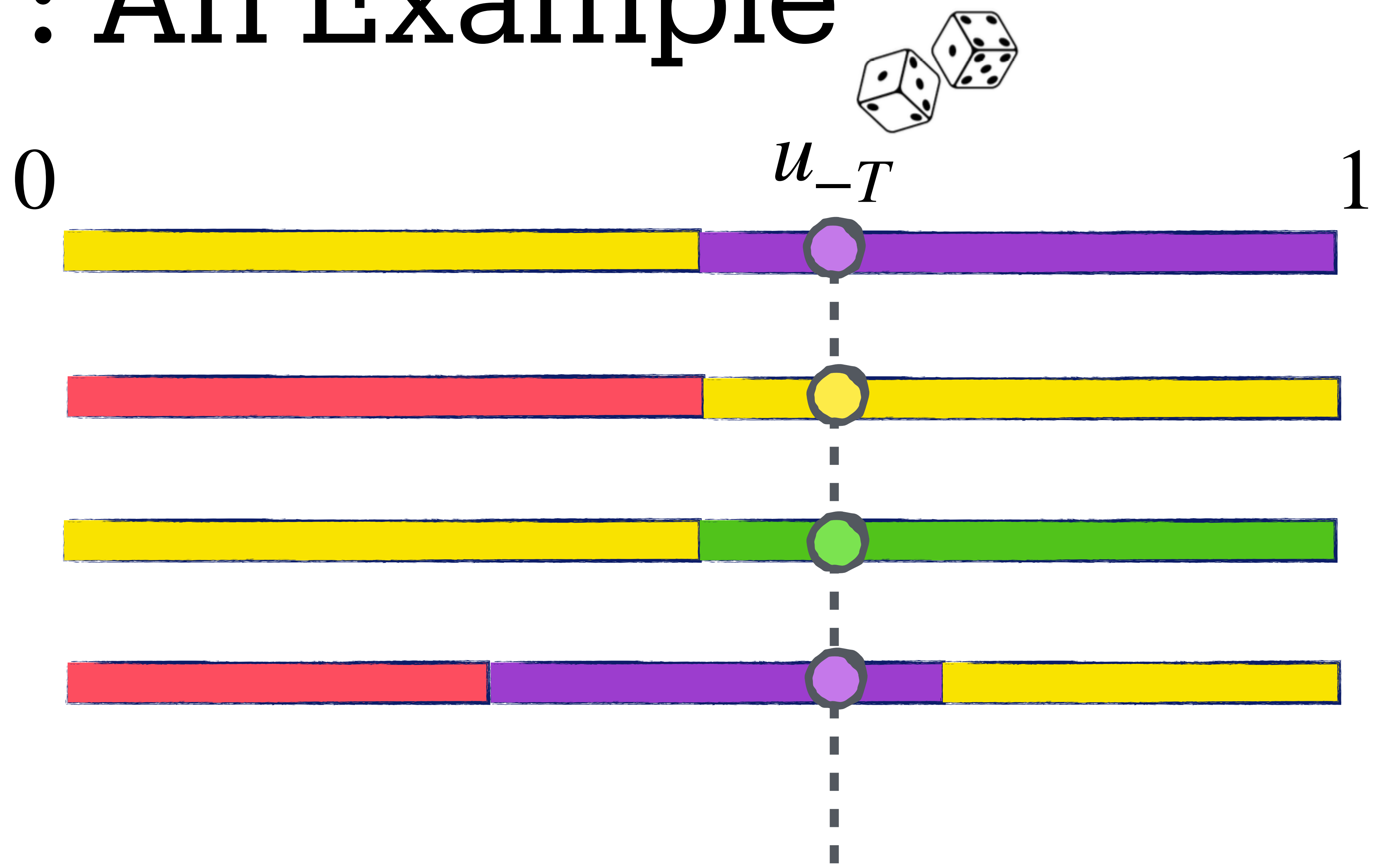
$\mathscr{P}_v$

Uniform{● ●}

Uniform{● ●}

Uniform{● ●}

Uniform{●●●}



$\mathscr{L}_{-T}(v) = \{● ● ● ●\}$ $\xrightarrow{g_{-T}}$ $\mathscr{L}_{-T+1}(v) = \{● ● ●\}$

Neighbors have bounding lists

$$\mathscr{L}_v = \{u \in \Gamma(v), \mathscr{L}(u)\}$$

They induce a family:

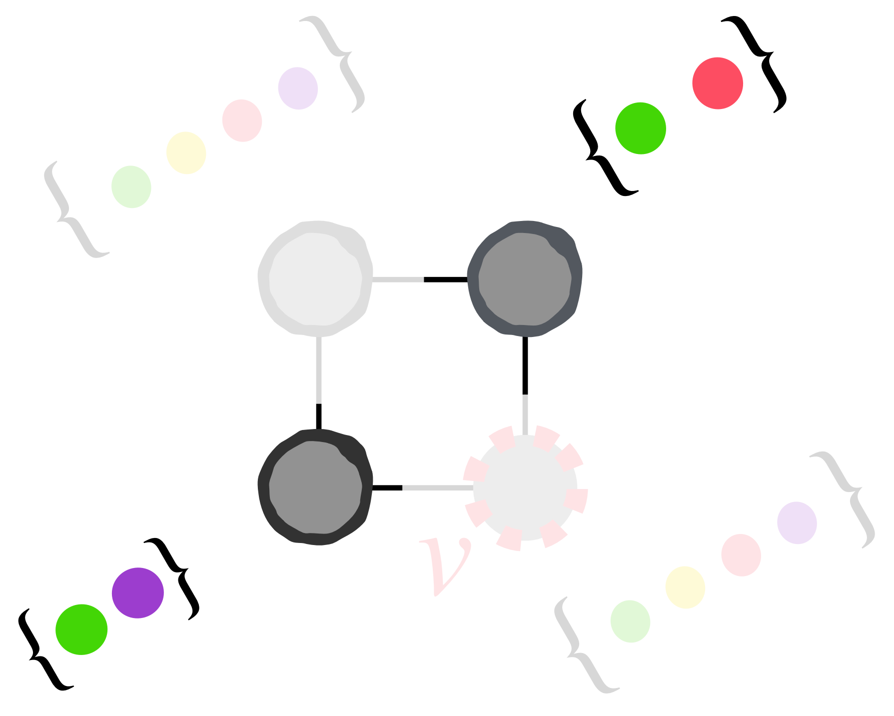
$$\mathscr{P}_v = \{\text{Uniform}([q]\backslash\sigma_v) : \sigma_v \in \mathscr{L}_v\}$$

Fixing the seed u_{-T} makes each distribution output exact one deterministic color.

Collect all possible outputs to form the updated list $\mathscr{L}_{-T+1}$.

Local Grand Coupling : An optimizing problem

Update v



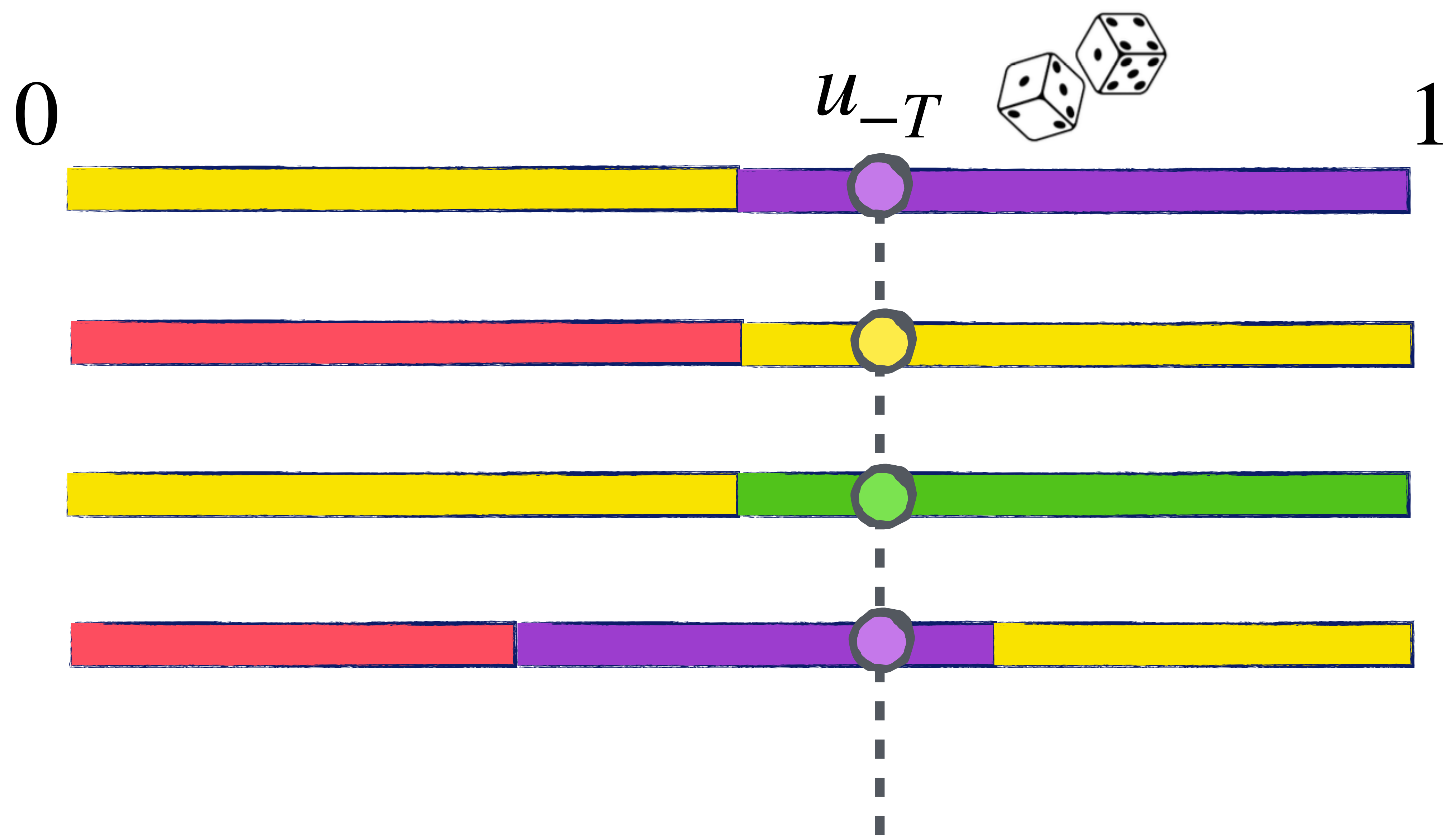
P_v

Uniform{● ●}

Uniform{● ●}

Uniform{● ●}

Uniform{●●●}



$\mathscr{L}_{-T}(v) = \{●●●●\} \xrightarrow{g_{-T}} \mathscr{L}_{-T+1}(v) = \{●●●\}$

Local Grand Coupling

A local grand coupling is a map

$$g_v : \mathscr{P}_v \times [0,1] \to [q]$$

such that for every $p \in \mathscr{P}_v$

$$\Pr_R[g_v(p, U) = a] = p(a), \qquad \forall a \in [q] .$$

Optimizing Problem

Define

$$L'_U(v) = \{g_v(p, U) : p \in \mathscr{P}_v\} .$$

Goal: Choose g_v to minimize

$$\mathbb{E}_U[\,|L'_U(v)|\,] .$$

Optimal Coupling via LP

Optimal Local Grand Coupling

Given neighborhood color

$$\mathcal{S} = \bigcup_{u \in \Gamma(v)} \mathcal{L}(u), \quad \mathcal{S}_\Delta = \{C \in S : |C| \le \Delta\}$$

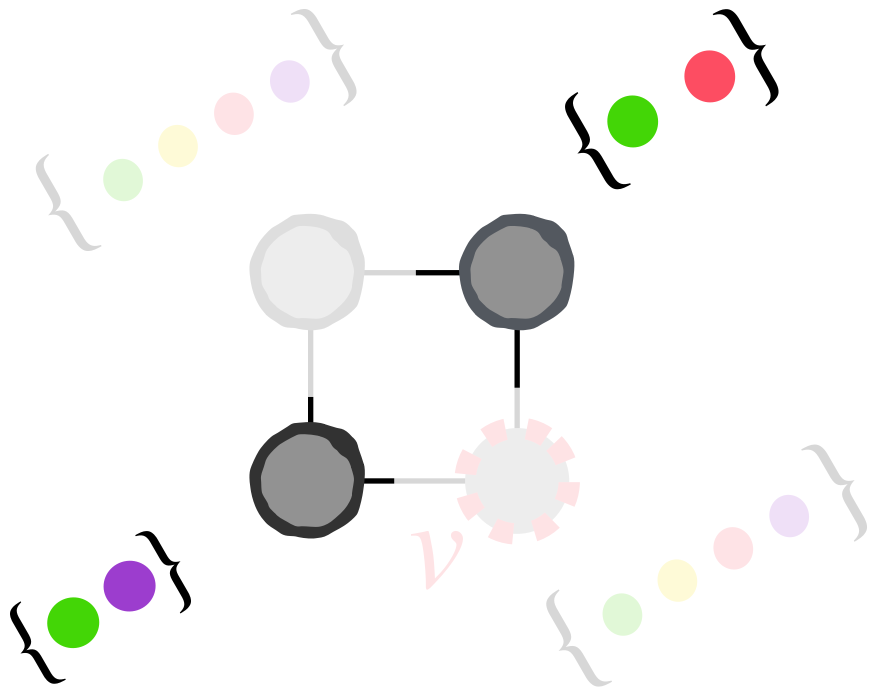
Let induced distribution family be

$$\mathcal{P}_\Delta = \{\text{Unifrom}([q] \backslash C) : C \in S_\Delta\}$$

Design local grand coupling g to minimizing the expectation of the output bounding list (only $\mathcal{S}$ is known)

$$L_\Delta(g, u) := |g(P, u) : P \in \mathcal{P}_\Delta|$$

$$\operatorname*{argmin}_g \mathbb{E}_u \left[L_\Delta(g, u) | \right]$$



$$\mathcal{S} = \{\bullet \bullet \bullet\}$$

$$T = [q]/S = \{ \bullet \}$$

We look at all $C \subseteq S$ with $|C| \le 2$

It is a relaxed version of
the real bounding-chain update.

Optimal Coupling via LP

Optimal Local Grand Coupling

Given neighborhood color

$\mathcal{S} = \bigcup_{u\in\Gamma} \mathcal{L}(u)$ … $|S| \le \Delta\}$

Find

- Nece…
  every…
- Suff…iency:
  every feasible r_k can be realized.
  solution → actual optimal local coupling

The LP has an explicit solution

$$r_{i-1} = \frac{w - z_{\Delta,i}}{z_{\Delta,i-1} - z_{\Delta,i}}, \quad , r_i = \frac{z_{\Delta,i-1} - w}{z_{\Delta,i-1} - z_{\Delta,i}}$$

For some i, w.

LP

Let r_k be the fraction of u that "hit" exact k colors. Let

$$z_{j,k} = \binom{j}{k-1} \Big/ \binom{|S|}{k-1}$$

Consider the linear program

$$\min \sum_{k=1}^{\Delta} k r_k \quad \text{s.t.}$$

$$\sum_{k=1}^{\Delta} r_k = 1$$

$$\sum_{k=1}^{\Delta} r_k z_{j,k} \le \frac{q - |S|}{q - j} \qquad j = 1,2,\cdots,\Delta$$

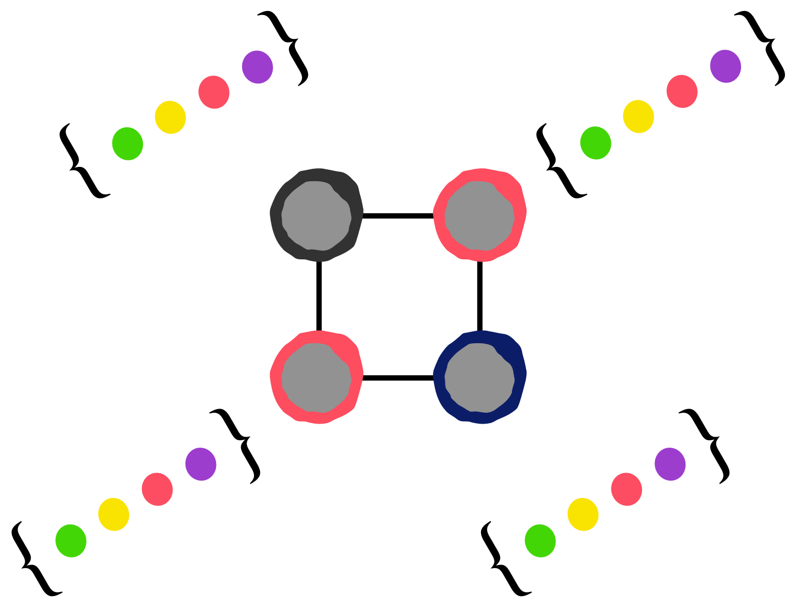
$$r_k \ge 0 \qquad k = 1,2,\cdots,\Delta$$

Our Algorithm

Improving [JSS21] via optimal local grand coupling

Mark $1/2 \pm o(1)$ as seeding set by LLL.

Phase I



Initially $\forall v, \mathscr{L}(v) = [q]$

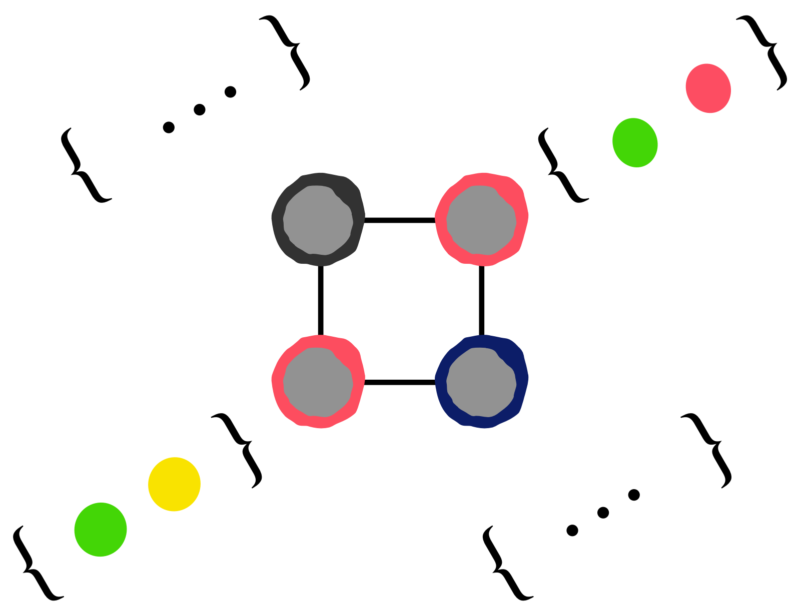
Scan seeding set

Random difting $O(n \log n)$ steps

Applying optimal coupling(new seeding)

require $q \geq 2.5\Delta$

Compress every neighbor before seeding update



For every , $|\mathscr{L}(v)| \leq 2$

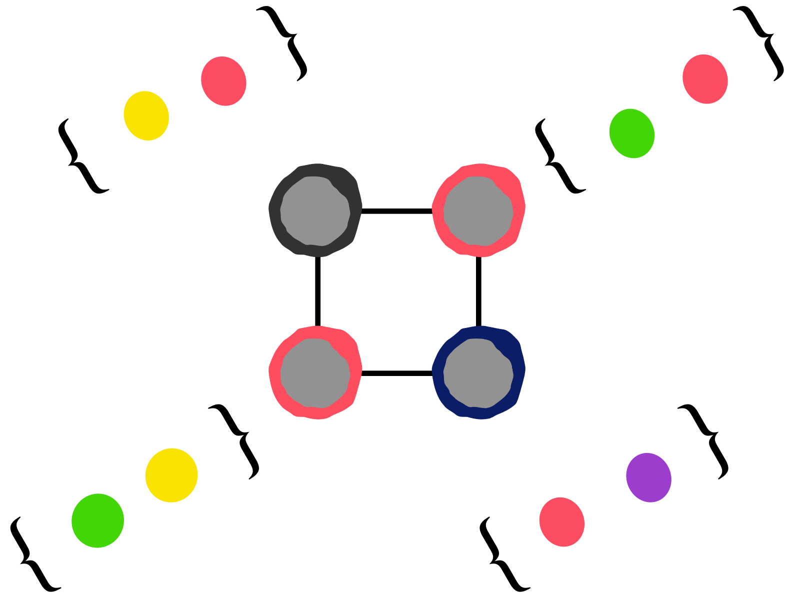
Phase II

Scan except seeding set

Disjoint coupling

require $q \geq 2.5\Delta$

(by refining argument)

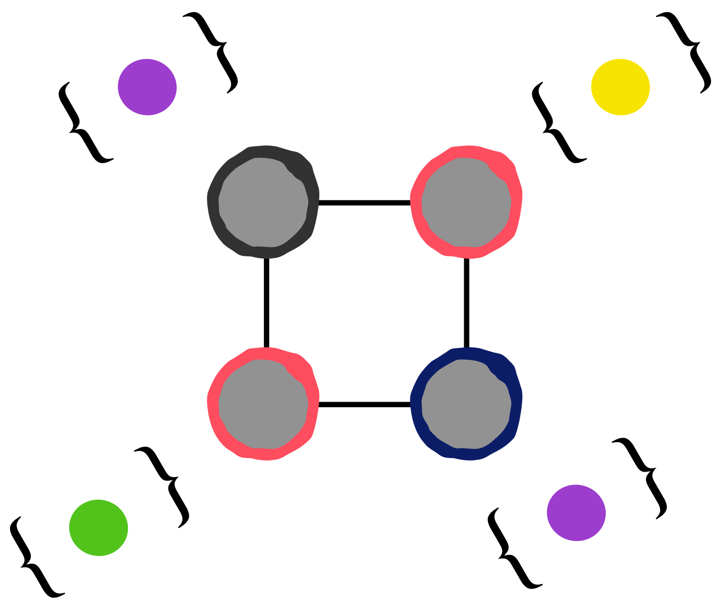


$|\mathscr{L}(v)| \leq 2$ For all vertices

Random difting $O(n \log n)$ steps

Disjoint coupling

Proving 2.5Δ lower bound here



$|\mathscr{L}(v)| = 1$ For all vertices

Comparison with [JSS21]

	[JSS21]	Our work	Our contribution
Phase I (seeding)	$q \geq \left(\frac{8}{3} + o(1)\right)\Delta$	$q \geq \left(2.5 + o(1)\right)\Delta$	optimal local coupling construction via LP solution
Phase II (disjoint)	$q \geq \frac{8}{3}\Delta$	$q \geq \left(2.5 + o(1)\right)\Delta$	proving 2.5Δ as lower-bound bottleneck

Lower Bound: the 2-to-1 Bottleneck

Decay condition for bounding chains

$$\mathbb{E}\left[\sum_u |L'(u)| \mid L\right] \le \sum_u |L(u)|$$

2-to-1 bottleneck

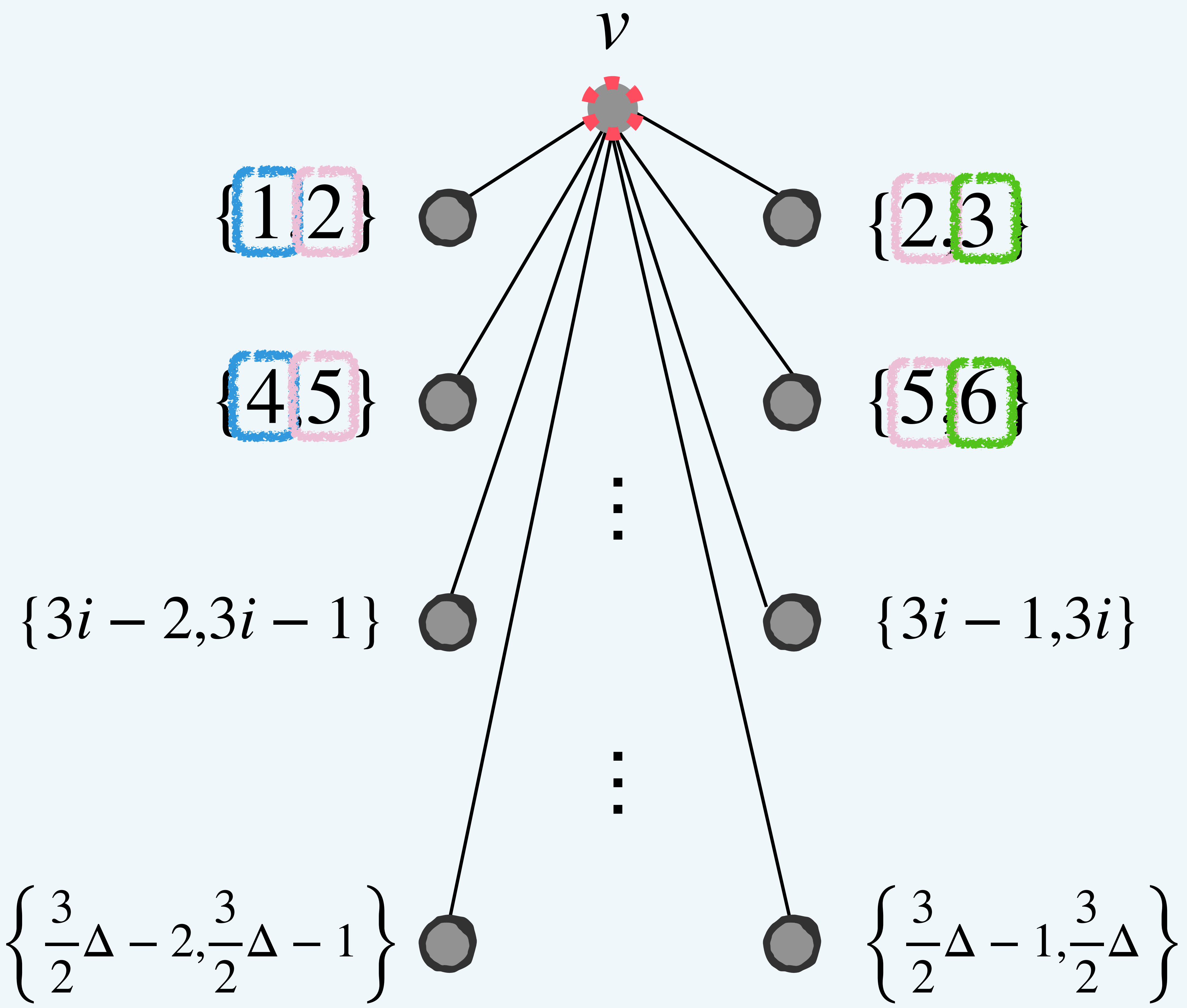
Shrinking 2-lists to singletons is the hard step.

below 2.5Δ

Every update increases the expected list size.

Contraction fails.

Worst-case triangle configuration



Summary

Problem: perfect sampling of q-colorings via bounding-chain CFTP.

Tight threshold: $q = (2.5 + o(1))\Delta$.

Framework:

CFTP $\longrightarrow$ grand couplings;
bounding chains $\longrightarrow$ local grand coupling.

Upper bound:

LP-based optimal coupling.

Lower bound:

worst-case barrier for contractive bounding chains.

Open Problems

- Toward perfect sampling below 2.5Δ beyond bounding chains.
- Universal barriers for CFTP / grand-coupling-based perfect sampling.
- Understanding and developing grand couplings as algorithmic tools.

Thanks! Questions welcome!